ON THE CONSTRUCTION OF LIAPUNOV'S FUNCTIONS FOR SYSTEMS OF LINEAR FINITE DIFFERENCE EQUATIONS WITH VARIABLE COEFFICIENTS

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In this paper there is presented one of several possible methods of the construction of Liapunov's function for a system of linear difference equations with variable coefficients, and of deriving sufficient conditions for asymptotic stability of the trivial solution of the mentioned systems. The proposed method is an extension of the author's work [1] which was a study of systems of linear differential equations with variable coefficients. This method permits one to determine the region of the space of the coefficients within which the functions (which are the variable coefficients of the considered system of equations) may vary without violation of the stability of the system.

Problems in which the law of variation in time of the variable coefficients is not known in advance but in which there are known only the boundaries within which these coefficients are enclosed, occur quite frequently in applications [2].

1. Let us consider the system of difference equations

$$x_j(t+\tau)+\sum_{k=1}^{n} b_{jk}(t)\,x_k(t)=0 \qquad (j=1,\ldots,n) \tag{1.1}$$

Introducing the functions

$$l_{jk}(t)=b_{jk}(t)+a_{jk} \qquad (a_{jk}=\mathrm{const}) \tag{1.2}$$

where a_{jk} are constants to be determined lated, the system (1.1) may be reduced to the form

$$x_j(t+\tau)=\sum_{k=1}^{n} a_{jk}x_k(t)-\sum_{k=1}^{n} l_{jk}(t)\,x_k(t) \qquad (j=1,\ldots,n) \tag{1.3}$$

Along with the system (1.3), we shall also consider the following system of nonhomogeneous difference equations with constant coefficients:

$$x_j(t+\tau)=\sum_{k=1}^{n} a_{jk}x_k(t)+y_j(t) \qquad (j=1,\ldots,n) \tag{1.4}$$

The system of scalar equations (1.4) is equivalent to the matrix equation

$$f(T)\,x(t) = y(t) \qquad (f(T) = ET - a) \tag{1.5}$$

Here

$$x(t) = \|x_j(t)\|, \qquad a = \|a_{jk}\|, \qquad y(t) = \|y_j(t)\| \tag{1.6}$$

E is the unit matrix, and T denotes the translation operator defined by

$$Tx(t) = x(t+\tau) \qquad (\tau = \text{const}) \tag{1.7}$$

The determinant of the matrix $f(T)$ is

$$\det f(T) = \begin{vmatrix} T - a_{11} & -a_{12} & \ldots & -a_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ -a_{n1} & -a_{n2} & \ldots & T - a_{nn} \end{vmatrix} \tag{1.8}$$

The roots of the characteristic equation

$$\det f(\gamma) = 0 \tag{1.9}$$

will be denoted as follows: the real roots by $\varkappa_g$ ($g = 1, \ldots, s'$), the complex roots by $\alpha_h \pm \beta_h i$ ($h = 1, \ldots, s''$). The number of roots of the characteristic equation is $n = s' + 2s''$. We shall assume that all roots of the characteristic equation (1.9) are simple roots. This can be achieved by a proper choice of the matrix a.

Below we shall need the general solution of the matrix equation

$$Tx - ax = 0 \tag{1.10}$$

Denoting by $\sim$ the integral part of t/τ, one can express the general solution of Equation (1.10) in the form

$$x = \sum_{g=1}^{s'} v_g A_g(t) \varkappa_g^{\vartheta} + \sum_{h=1}^{s''} \{V_{s'+h}\,[A_{s'+h}(t) + iA_{s'+s''+h}(t)]\,(\alpha_h + \beta_h i)^{\vartheta} +$$

$$+ V_{s'+s''+h}\,[A_{s'+h}(t) - iA_{s'+s''+h}(t)]\,(\alpha_h - \beta_h i)^{\vartheta}\} \tag{1.11}$$

Here, v_g is a nonzero column of the matrix

$$F(\varkappa_g) = [F(T)]_{T=\varkappa_g}$$

where $F(T)$ is the adjoint matrix of the matrix $f(T)$. By $V_{s'+h}$ we denote the nonzero column of the matrix $F(\alpha_h + \beta_h i)$, and by $V_{s'+s''+h}$ the same column of $F(\alpha_h - \beta_h i)$. The similarly located elements of the matrix columns $V_{s'+h}$ and $V_{s'+s''+h}$ are therefore complex conjugates.

The function $A_k(t)$ ($k = 1, \ldots, n$) which occur in the expression (1.11) are arbitrary real periodic functions of period τ. The form of these functions can be determined by means of the solution of Equation (1.5) if one specifies the functions $x_j(t)$ ($j = 1, \ldots, n$) on the interval of time $0 < t < \tau$.

We note that since by hypothesis all roots of the characteristic equation (1.9) are simple, the ranks of the matrices $F(\varkappa_g)$ and $F(\alpha_h + \beta_h i)$ are equal to one, i.e. the nonzero column of each of these matrices are proportional to each other. For the sake of convenience one can normalize the matrix columns $v_g, V_{s'+h}, V_{s'+s''+h}$, which appear in Expression (1.11), by dividing each element of each of these marices by one of its nonzero elements. This leads to the following relation:

$$v_g B_g = F(\varkappa_g) \tag{1.12}$$

Here B_g is a row of the matrix $F(\varkappa_g)$ which contains the element, the division by which led to the column v_g. Analogously,

$$V_{s'+h} B_{s'+h} = F(\alpha_h + \beta_h i), \qquad V_{s'+s''+h} B_{s'+s''+h} = F(\alpha_h - \beta_h i) \tag{1.13}$$

where the corresponding elements of the matrix rows $B_{s'+h}$ and $B_{s'+s''+h}$ are complex conjugate quantities.

Let us denote by r the compounded matrix row whose elements are the matrix columns of (1.11),

$$r = \| v_1 \dots v_{s'} \quad V_{s'+1} \dots V_{s'+s''} \quad V_{s'+s''+1} \dots V_{s'+2s''} \| \tag{1.14}$$

We introduce the matrix $\Xi = \| \Xi_j \| \, (j = 1, \dots, n)$ with the aid of the linear transformation

$$x = r\Xi \tag{1.15}$$

Since $\Xi\,(t+\tau) = r^{-1}x\,(t+\tau)$, we have in accordance with (1.5) the relation

$$T\Xi = k\Xi + r^{-1}y\,(t) \qquad (k = r^{-1}ar) \tag{1.16}$$

Here the matrix k has the following form

$$k = \begin{Vmatrix} \varkappa & 0 & 0 \\ 0 & \alpha + \beta i & 0 \\ 0 & 0 & \alpha - \beta i \end{Vmatrix}, \qquad \varkappa = \begin{Vmatrix} \varkappa_1 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & \varkappa_{s'} \end{Vmatrix}$$

$$\alpha \pm \beta i = \begin{Vmatrix} \alpha_1 \pm \beta_1 i & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & \alpha_{s''} \pm \beta_{s''} i \end{Vmatrix} \tag{1.17}$$

The relation (1.17) can be verified in the following way. Substituting into Equation (1.10) the particular solution $x = r_\mu A_\mu^\vartheta(t)\gamma_\mu^\vartheta \; (\mu = 1, \dots, n)$, where r_μ is a column of the matrix (1.14) while γ_μ is a root of the characteristic equation (1.9), we find that $(E\gamma_\mu - a) r_\mu A_\mu (t)\gamma_\mu^\vartheta = 0$. Hence, $ar_\mu = r_\mu \gamma_\mu$. Since the matrix k is a diagonal matrix in which $k_{jj} = \gamma_j$, $k_{jl} = 0 \; (j \neq l)$, it follows from the preceding relation that $ar = rk$; and, since r is a nonsingular matrix, then $r^{-1}ar = k$.

The matrix equation (1.16) is equivalent to the following system of scalar equations:

$$T\Xi_g = \varkappa_g \Xi_g + Y_g(t) \qquad (g = 1, \dots, s')$$

$$T\Xi_{s'+h} = (\alpha_h + \beta_h i)\Xi_{s'+h} + Y_{s'+h}(t)$$

$$T\Xi_{s'+s''+h} = (\alpha_h - \beta_h i)\,\Xi_{s'+s''+h} + Y_{s'+s''+h}(t) \qquad (h = 1, \dots, s'') \tag{1.18}$$

where

$$Y\,(t) = r^{-1}y\,(t) \tag{1.19}$$

Let us introduce new variables

$$\xi_g \;\; (g = 1, \dots, s'), \; \eta_h, \;\; \zeta_h \;\; (h = 1, \dots, s'')$$

with the aid of the relations

$$\Xi_g = \xi_g \qquad (g = 1, \dots, s')$$

$$\Xi_{s'+h} = {}^1/_2 (\eta_h - i\zeta_h), \qquad \Xi_{s'+s''+h} = {}^1/_2 (\eta_h + i\zeta_h) \qquad (h = 1, \dots, s'') \tag{1.20}$$

From the relations (1.14) and (1.19) it can be seen that the elements of the matrix columns $Y_{s'+h}(t)$ and $Y_{s'+s''+h}(t)$ are complex conjugates. With this in mind, one can, in accordance with (1.18), represent the difference equations which are satisfied by the new variables in the form

$$T\xi_g = \varkappa_g \xi_g + \sum_{k=1}^{n} (r^{-1})_{gk} y_k(t) \qquad (g = 1, \dots, s')$$

$$T\eta_h = \alpha_h \eta_h + \beta_h \zeta_h + 2\,\mathrm{Re} \sum_{k=1}^{n} (r^{-1})_{s'+h,\,k} y_k(t)$$

$$T\zeta_h = -\beta_h \eta_h + \alpha_h \zeta_h - 2\,\mathrm{Im} \sum_{k=1}^{n} (r^{-1})_{s'+h,\,k} y_k(t) \qquad (h = 1, \dots, s'') \tag{1.21}$$

Let us return now to the original matrix equation (1.3). In order that the system of equations (1.21) may be equivalent to the matrix difference equation (1.3), it is necessary to assume, in accordance with (1.4), that

$$y_k(t) = -\sum_{\sigma=1}^{n} l_{k\sigma}(t)\, x_\sigma(t) \qquad (k = 1, \ldots, n) \tag{1.22}$$

It is still necessary to express the x_k, in the relation (1.22), in terms of the new variables ξ_g, η_h, ζ_h. The matrix relation (1.15) can be presented, according to (1.14), in the form

$$x = \sum_{g=1}^{s'} v_g \Xi_g + \sum_{h=1}^{s''} (V_{s'+h}\Xi_{s'+h} + V_{s'+s''+h}\Xi_{s'+s''+h}) \tag{1.23}$$

Taking into account the fact that the elements of the matrix columns $V_{s'+h}$ and $V_{s'+s''+h}$ are complex conjugate, and using the notation

$$V_{s'+h},\ V_{s'+s''+h} = v_{s'+h} \pm i v_{s'+s''+h} \tag{1.24}$$

we obtain, corresponding to (1.20),

$$x = \sum_{g=1}^{s'} v_g \xi_g + \frac{1}{2}\sum_{h=1}^{s''} [(v_{s'+h} + i v_{s'+s''+h})(\eta_h - i\zeta_h) + (v_{s'+h} - i v_{s'+s''+h})(\eta_h + i\zeta_h)] \tag{1.25}$$

or

$$x = \sum_{g=1}^{s'} v_g \xi_g + \sum_{h=1}^{s''} (v_{s'+h}\eta_h + v_{s'+s''+h}\zeta_h) \tag{1.26}$$

The elements of the matrix x will be

$$x_\sigma = \sum_{g=1}^{s'} v_{g\sigma}\xi_g + \sum_{h=1}^{s''} (v_{s'+h,\,\sigma}\eta_h + v_{s'+s''+h,\,\sigma}\zeta_h) \qquad (\sigma = 1, \ldots, n) \tag{1.27}$$

Substituting in the right-hand side of Equation (1.21) in place of the functions $y_k(t)$ their values from (1.22) and (1.27), one can reduce Equations (1.21) to the form

$$\begin{aligned}
T\xi_g &= \varkappa_g \xi_g + \sum_{k=1}^{s'} \mu_{gk}(t)\,\xi_k + \sum_{\lambda=1}^{s''} [\nu_{g\lambda}(t)\,\eta_\lambda + \rho_{g\lambda}(t)\,\zeta_\lambda] \qquad (g = 1, \ldots, s') \\
T\eta_h &= \alpha_h \eta_h + \beta_h \zeta_h + \sum_{k=1}^{s'} \mu_{s'+h,\,k}(t)\,\xi_k + \sum_{\lambda=1}^{s''} [\nu_{s'+h,\,\lambda}(t)\,\eta_\lambda + \rho_{s'+h,\,\lambda}(t)\,\zeta_\lambda] \\
T\zeta_h &= -\beta_h \eta_h + \alpha_h \zeta_h + \sum_{k=1}^{s'} \mu_{s'+s''+h,\,k}(t)\,\xi_k + \\
&\quad + \sum_{\lambda=1}^{s''} [\nu_{s'+s''+h,\,\lambda}(t)\,\eta_\lambda + \rho_{s'+s''+h,\,\lambda}(t)\,\zeta_\lambda] \qquad (h = 1, \ldots, s'')
\end{aligned} \tag{1.28}$$

The system of equations (1.20) is equivalent to the original equation (1.1).

We note now, that in the transition from the original equation (1.1) to the equation (1.3) no restrictions were laid on the coefficients a_{jk}; it was only required the the relations (1.2) hold. Below we shall show that the coefficients a_{jk} be best chosen so that all the roots of the characteristic equation (1.9) might be simple and that they satisfy the conditions

$$|\varkappa_g| < 1 \quad (g = 1, \ldots, s'), \qquad |\alpha_h + \beta_h i| < 1 \quad (h = 1, \ldots, s'') \tag{1.29}$$

If these conditions are satisfied, the solutions of the homogeneous equation (1.10) will tend asymptotically to zero when $t \to \infty$.

2. The system of equations (1.28) is also a system of linear difference equations with variable coefficients similar to the original system (1.1). However, for the system (1.28) one can give a simple method of construction of Liapunov's function which permits one to obtain sufficient conditions for the stability of the trivial solution of this system.

Let us take the Liapunov function in the form

$$V = -\sum_{g=1}^{s'} \xi_g^2 - \sum_{h=1}^{s''} (\eta_h^2 + \zeta_h^2) \tag{2.1}$$

The function V is negative-definite. Its first difference

$$\Delta V = -\sum_{g=1}^{s'} [\xi_g^2 (t+\tau) - \xi_g^2 (t)] - \\ - \sum_{h=1}^{s''} [\eta_h^2 (t+\tau) - \eta_h^2 (t) + \zeta_h^2 (t+\tau) - \zeta_h^2 (t)] \tag{2.2}$$

takes the following form, after the substitution of the values $\xi_g (t+\tau)$, $\eta_h (t+\tau)$, $\zeta_h (t+\tau)$ from Equations (1.28)

$$\Delta V = \sum_{g=1}^{s'} [(1-\varkappa_g^2)\xi_g^2 - 2\varkappa_g \xi_g \Lambda_g - \Lambda_g^2] + \\ + \sum_{h=1}^{s''} \{[1-(\alpha_h^2+\beta_h^2)](\eta_h^2+\zeta_h^2) - 2[(\alpha_h \Lambda_{s'+h} - \beta_h \Lambda_{s'+s''+h})\eta_h + \\ + (\beta_h \Lambda_{s'+h} + \alpha_h \Lambda_{s'+s''+h})\zeta_h] - \Lambda_{s'+h}^2 - \Lambda_{s'+s''+h}^2\} \tag{2.3}$$

where by Λ_j $(j = 1, \ldots, n)$ we denote the functions

$$\Lambda_j = \sum_{k=1}^{s'} \mu_{jk}(t)\xi_k + \sum_{\lambda=1}^{s''} [\nu_{j\lambda}(t)\eta_\lambda + \rho_{j\lambda}(t)\zeta_\lambda] \qquad (j = 1, \ldots, n) \tag{2.4}$$

which occur in the right-hand side of Equations (1.28).

Equations (2.3) and (2,4) show that the function ΔV is a quadratic form in the variables ξ_g, η_h, ζ_h of the form

$$\Delta V = \sum_{g=1}^{s'} [1-\varkappa_g^2 + \Psi_g(t)]\xi_g^2 + \\ + \sum_{h=1}^{s''} \{[1-(\alpha_h^2+\beta_h^2) + \Psi_{s'+h}(t)]\eta_h^2 + [1-(\alpha_h^2+\beta_h^2) + \Psi_{s'+s''+h}(t)]\zeta_h^2\} + \\ + 2c_{12}\xi_1\xi_2 + 2c_{13}\xi_1\xi_3 + \ldots + 2c_{1,\, s'+s''}\xi_1\eta_{s''} + \ldots + 2c_{1n}\xi_1\zeta_{s''} + \\ + 2c_{23}\xi_2\xi_3 + \ldots + 2c_{2,\, s'+s''}\xi_2\eta_{s''} + \ldots + 2c_{2n}\xi_2\zeta_{s''} + \ldots + 2c_{n-1,\, n}\zeta_{s''-1}\zeta_{s''} \tag{2.5}$$

The coefficients c_{ij} $(i \neq j)$ of the quadratic form (2.5), as well as the functions $\Psi_j(t)$ $(j = 1, \ldots, n)$, are combinations of the original variable coefficients $l_{jk}(t)$.

In the particular case when the original coefficients $l_{jk}(t) \equiv 0$ $(j, k = 1, \ldots \ldots, n)$, the function (2.5) take the form

$$\Delta V = \sum_{g=1}^{s'} (1-\varkappa_g^2)\xi_g^2 + \sum_{h=1}^{s''} [1-(\alpha_h^2+\beta_h^2)](\eta_h^2+\zeta_h^2) \tag{2.6}$$

If the roots of the characteristic equation (1.9) satisfy the condition (1.29)

$$|\varkappa_g| < 1 \quad (g = 1, \ldots, s'), \qquad |\alpha_h + \beta_h i| < 1 \quad (h = 1, \ldots, s'')$$

then the function (2.6) is positive-definite and the sign is the opposite of the sign of the Liapunov function V.

If the conditions (1.29) are fulfilled, the solution (1.11) of the matrix equation (1.10) will tend asymptotically to zero as $t \to \infty$, i.e. the trivial solution of Equation (1.10) is asymptotically stable.

In the general case, when the functions $l_{jk}(t) \not\equiv 0$ $(j, k = 1, \ldots, n)$, it is necessary to consider the quadratic form (2.5). Its discriminant has the form

$$D = \begin{vmatrix} c_{11} & \cdots & c_{1n} \\ \cdots & \cdots & \cdots \\ c_{n1} & \cdots & c_{nn} \end{vmatrix} \tag{2.7}$$

where

$$c_{gg} = 1 - \varkappa_g^2 + \Psi_g(t) \qquad (g = 1, \ldots, s')$$

$$c_{s'+h,\, s'+h} = 1 - (\alpha_h^2 + \beta_h^2) + \Psi_{s'+h}(t) \tag{2.8}$$

$$c_{s'+s''+h,\, s'+s''+h} = 1 - (\alpha_h^2 + \beta_h^2) + \Psi_{s'+s''+h}(t) \qquad (h = 1, \ldots, s'')$$

If for every $t > 0$ all principal diagonal minors of the discriminant (2.7) are not less than some positive number, then the first difference ΔV of Liapunov's function (2.1) will be a positive-definite function and its sign will be opposite to the sign of Liapunov's function.

It is shown in [3], where an extension of Liapunov's theorem to equations for finite differences is given, that this property of the first difference of Liapunov's function is a sufficient condition for the asymptotic stability of the trivial solution of a system of equations in finite differences.

Thus, the conditions of strong positiveness at any time $t > 0$ of all principal diagonal minors of the discriminant (2.7) are sufficient conditions for the asymptotic stability of the trivial solution of the original system (1.1) of difference equations with variable coefficients.

Since the obtained conditions for stability are sufficient but not necessary, one can sometimes, by varying the form of the function V, broaden the region of stability obtained from the condition of the sign-definiteness of ΔV in the space of the parameters of the system. In order to be able to vary the function V, it may be taken in the form

$$V = -\sum_{g=1}^{s'} p_g \xi_g^2 - \sum_{h=1}^{s''} (p_{s'+h} \eta_h^2 + p_{s'+s''+h} \zeta_h^2) \tag{2.9}$$

where the coefficients p_j $(j = 1, \ldots, n)$ must be strongly positive. The choice of the coefficients p_j can be subjected to definite requirements; for example, certain terms of the principal diagonal minor of the discriminant (2.7) may be made to vanish.

The choice of the coefficients a_{jk} which appear in the expression (1.2) should also be subjected to the requirement of the maximal broadening of the region of stability in the space of the parameters of the system.

3. As an example, let us consider the following system of difference equations:

$$Tx_1 - x_1 - kx_2 = 0, \qquad Tx_2 + m(t)x_1 - (1 - \lambda k)x_2 = 0 \tag{3.1}$$

In Equations (3.1) the coefficients λ and k are assumed to be real positive.

The system (3.1) can be put into the form

$$Tx_1 - x_1 - kx_2 = 0, \qquad Tx_2 + \sigma x_1 - (1 - \lambda k)x_2 = g(t)x_1 \qquad (g(t) = \sigma - m(t)) \tag{3.2}$$

Here the coefficient σ in the second equation of (3.2) is chosen so that for the system of different equations

$$Tx_1 - x_1 - kx_2 = 0, \qquad Tx_2 + \sigma x_1 - (1 - \lambda k)x_2 = 0$$

the characteristic equation

$$\gamma^2 - (2 - \lambda k)\gamma + 1 - \lambda k + k\sigma = 0$$

may have a pair of complex roots $\gamma_1, \gamma_2 = \alpha \pm \beta i$, whose absolute value

$$(\alpha^2 + \beta^2)^{1/2} < 1$$

In order that this may happen, the coefficient σ must satisfy the conditions

$$\sigma > {}^1/_4\lambda^2 k, \qquad 0 < 1 - \lambda k + k\sigma < 1$$

Let us now go over to the new variables η and ζ which, in accoedance with (1.27), are introduced with the aid of the relations

$$x_1 = \eta, \qquad x_2 = -\frac{1-\alpha}{k}\eta + \frac{\beta}{k}\zeta \tag{3.3}$$

The difference equations which are satisfied by η and ζ, in accordance with (3.2) are

$$T\eta = \alpha\eta + \beta\zeta, \qquad T\zeta = [-\beta + k\beta^{-1}g(t)]\eta + \alpha\zeta \tag{3.4}$$

We take Liapunov's function, according to (2.1), in the form

$$V = -(\eta^2 + \zeta^2) \tag{3.5}$$

The first difference of the function V, by the difference equation (3.4), will be

$$\Delta V = c_{11}\eta^2 + 2c_{12}\eta\zeta + c_{22}\zeta^2 \tag{3.6}$$

where

$$c_{11} = 1 - \alpha^2 - [\beta - k\beta^{-1}g(t)]^2, \qquad c_{12} = -k\alpha\beta^{-1}g(t), \qquad c_{22} = 1 - (\alpha^2 + \beta^2) \tag{3.7}$$

The sufficient conditions for asymptotic stability of the trivial solution of the system of the difference equations (3.1) are that for any $t > 0$ the principal diagonal minors of the discriminant of the quadratic form (3.6)

$$D_1 = c_{11}(t), \qquad D_2 = c_{11}(t)\,c_{22}(t) - [c_{12}(t)]^2 \tag{3.8}$$

must be not less than some positive number.

For the given values of the system's parameters λ and k one can determine from these conditions a strip

$$g_1 < g(t) < g_2$$

within which the function $g(t)$ can vary arbitrarily, without violation of the stability of the system.

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